

Linear Phase Perfect Reconstruction Filters and Wavelets with Even Symmetry.

Lucas Monzón

Department of Applied Mathematics

University of Colorado

Boulder, CO 80309-0526

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Abstract

Perfect reconstruction filter banks can be used to generate a variety of wavelet bases. Using IIR linear phase filters one can obtain symmetry properties for the wavelet and scaling functions.

In this paper we describe all possible IIR linear phase filters generating symmetric wavelets with any prescribed number of vanishing moments. In analogy with the well known FIR case, we construct and study a new family of wavelets obtained by considering maximal number of vanishing moments for each fixed order of the IIR filter. Explicit expressions for the coefficients of numerator, denominator, zeroes, and poles are presented.

This new parameterization allows one to design linear phase quadrature mirror filters with many other properties of interest such as filters that have any preassigned set of zeroes in the stopband or that satisfy an almost interpolating property.

Using Beylkin's approach, it is indicated how to implement these IIR filters not as recursive filters but as FIR filters.

Key words. Symmetric wavelets, orthonormal bases, linear phase filters

AMS subject classifications. 42, 42C40

1 Introduction

Quadrature mirror filters (QMFs)¹ allow one to design two-channel perfect reconstruction filter banks and therefore to generate wavelet bases. For implementation reasons we either consider finite impulse response (FIR) or realizable infinite impulse response (IIR) filters. The first class corresponds to polynomial filters, the second, wider class, to rational filters. In both cases, we consider filters with real coefficients and we identify filters with their z-transforms. Rational QMFs admit properties difficult or impossible to achieve with polynomial QMFs. For example, neither linear phase (except for the Haar filter) nor interpolation properties can be exactly obtained even though both properties can be approximately achieved using FIR coiflets [14]. In contrast, both properties can be exactly obtained with IIR filters but not simultaneously. Parameterizations of polynomials [15, 22] or rational QMFs [9] are well known. In both cases, linear conditions on the coefficients of the filter, such as vanishing moments for the wavelet or the scaling function, become non-linear conditions on the parameters describing the coefficients. This problem can sometimes be overcome, and there are explicit families of rational filters with an arbitrary number of zeroes at -1 , that is an arbitrary number of vanishing moments for the wavelet. But the additional requirement of symmetry conditions, which correspond to linear phase filters, again forces us to solve non-linear systems. For this reason, there are only some examples of linear phase filters generating wavelets with vanishing moments [10, 18].

Often, it is much simpler to describe the filters in terms of their absolute values (as in [3, 6] for polynomials or [10] for rational QMFs), leading to implicit constructions. In most cases the solutions are obtained numerically and the exact coefficients are not known except for small filter lengths. These numerical solutions are QMFs within a certain accuracy. In practical applications this is not an inconvenience because we always work within a certain precision that can in principle be attained except for large filter lengths. Nevertheless, we may like to incorporate the accuracy sought as a parameter in the process of design-implementation of the QMF.

Leaving aside the problem of accuracy once the size of the filter increases, we come across a much deeper problem. Namely, how does one simultaneously impose different properties in an exact or approximate sense? A good example is Daubechies' parameterization of the magnitude response of polynomial QMFs. Even though she used it to obtain the autocorrelation coefficients of maximally flat filters, it is not clear how to use this description to obtain other designs. For example, when obtaining coiflets, neither her method [8] nor the simpler description in [14] can prevent bad behavior of the

¹Originally, filters leading to perfect reconstruction were named conjugate quadrature filters [20] but we use the term QMF as in [7, Pages 162-163].

frequency response or guarantee zeroes in locations different than π . Theoretical results are also hard to obtain, and we cannot even establish the existence of coiflets for an arbitrary number of vanishing moments.

Here we propose a different approach for the design and implementation of QMFs. We first isolate a specific property of interest, like interpolation or linear phase, and then parameterize all possible rational QMFs with that property. The parameters can be used to impose additional properties or to optimize a particular design [13].

In this paper we choose linear phase as the initial property in our design. It implies two possible symmetries for the wavelet and scaling functions. Either the scaling function is even (and the wavelet is even about $1/2$) or it is even about $1/2$ (and the wavelet is odd about $1/2$). These two situations give rise to two different classes of linear phase filters that we denote 0-SYM and $1/2$ -SYM. In this paper we discuss the 0-SYM case. For the $1/2$ -SYM case see [10, 18, 13]. Even though there are no linear phase interpolating filters, it is possible to obtain linear phase and almost interpolating filters. It is enough to ask for a symmetric wavelet with enough vanishing moments. Both properties imply that the scaling function has vanishing moments (twice the number for the wavelet) and that assures the almost interpolating property [14].

Since linear phase filters have poles inside and outside the unit circle, we indicate how to implement these IIR by approximating them by cascades of FIR filters as in [1]. The FIR approximation does satisfy the quadrature mirror condition with any desired accuracy. The factors in the approximation directly depend on the location of the poles of the IIR filter, what allow one to introduce the accuracy as part of the filter design process.

The main result of this paper is the complete description, with closed form expressions, of all possible 0-SYM filters generating wavelets and scaling functions with symmetry properties and any prescribed number of vanishing moments. Each 0-SYM filter is described using a minimal set of parameters which are directly related with the zeroes of the filter. This property allow us to construct filters with any preassigned set of zeroes in the stopband. In particular, we construct, for each fixed order, the filters which generate wavelets with maximal number of vanishing moments. We also give sufficient conditions on the parameters to assure that the frequency responses of the filters are always positive or that the filters have all their poles on the imaginary axis.

The paper is organized as follows. In the next section we state some basic results and definitions and then identify the two kinds of symmetry. In Section 3 we describe, in closed form, all solutions of the 0-SYM case. We then identify a minimal set of parameters defining each solution, use them to obtain a variety of properties, and show explicit examples. The properties of the maximally flat

0-SYM filters and an example of its FIR implementation are discussed in Section 4.

2 Preliminaries

- Unless otherwise indicated, x and ξ are real variables while z is a complex variable.
- A QMF is a 2π -periodic function m_0 ,

$$m_0(\xi) = \sum_{k \in \mathbf{Z}} h_k e^{-ik\xi}, \quad (2.1)$$

such that

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1. \quad (2.2)$$

The numbers $\{h_k\}$ are the *coefficients* of the filter m_0 . We assume m_0 to be a rational real trigonometric function. That is, $\{h_k\}$ are real and have exponential decay, but maybe an infinite number of them is non-zero. Condition (2.2) is then equivalent to

$$2 \sum_k h_k h_{k+2n} = \delta_{no} \quad \text{for } n \in \mathbf{Z}, \quad (2.3)$$

where δ_{nm} is defined as $\delta_{nm} = 1$ if $m = n$, and $\delta_{nm} = 0$ otherwise.

We denote by H the z-transform of $\{h_k\}$, $H(z) = \sum_k h_k z^k$. We also refer to such H as a QMF. Its *frequency response* is $H(e^{-i\xi}) = m_0(\xi)$.

From (2.3), H satisfies the functional equation (the QMF equation),

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \quad (2.4)$$

In order to generate a regular Multiresolution Analysis (see [4]), we need two additional properties for H .

The first one is the *normalization* or *low-pass* condition. It forces

$$m_0(0) = 1 \quad \text{for } H(1) = 1. \quad (2.5)$$

The second one assures that H is non-zero in certain locations of the unit circle [5].

Cohen's condition There exist no non-trivial invariant cycles for $\xi \rightarrow 2\xi$ such that $|m_0| = 1$ on the whole cycle, i. e., there is no $\omega \neq 0$ with $2^n \omega = \omega \pmod{2\pi}$ such that for all $0 \leq j < n$

$$m_0(2^j \omega + \pi) = 0. \quad (2.6)$$

In practice, we first find a normalized H satisfying the QMF equation, and then verify Cohen's condition.

- A solution φ of

$$\varphi\left(\frac{x}{2}\right) = 2 \sum_k h_k \varphi(x - k) \quad (2.7)$$

is called a “scaling function”. Equivalently,

$$\hat{\varphi}(2\xi) = m_0(\xi) \hat{\varphi}(\xi), \quad (2.8)$$

where $\hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} \varphi(x) e^{-i\xi x} dx$ and m_0 is the *associated* filter.

From (2.5) and (2.8),

$$\hat{\varphi}(\xi) = \prod_{k=1}^{\infty} m_0(2^{-k}\xi) \quad \text{and} \quad \hat{\varphi}(0) = 1. \quad (2.9)$$

The *mother* wavelet ψ is defined on the Fourier side as

$$\hat{\psi}(2\xi) = m_1(\xi) \hat{\varphi}(\xi) \quad (2.10)$$

where m_1 is a 2π periodic function,

$$m_1(\xi) = e^{-i\xi} \lambda(\xi) \overline{m_0(\xi + \pi)}. \quad (2.11)$$

Here λ is π -periodic and $|\lambda(\xi)| = 1$ a.e. and $\overline{m_0}$ is the complex conjugate of m_0 .

2.1 The moment condition

One fundamental property for wavelet bases [2, 12, 14] is the property of vanishing moments of the wavelet or the scaling function:

$$\int_{\mathbf{R}} x^k \psi(x) dx = 0 \quad \text{for} \quad 0 \leq k < M, \quad (2.12)$$

$$\int_{\mathbf{R}} x^k \varphi(x) dx = \delta_{k0} \quad \text{for} \quad 0 \leq k < N. \quad (2.13)$$

In terms of the filter they become [7, Page 258]

$$D^k H(-1) = 0 \quad \text{for} \quad 0 \leq k < M, \quad (2.14)$$

$$D^k H(1) = \delta_{k0} \quad \text{for} \quad 0 \leq k < N. \quad (2.15)$$

2.2 The symmetry condition

We now find which filters generate symmetry conditions on the scaling and wavelet functions. For the scaling function to be symmetric, we need real constants λ and c such that for all x in \mathbf{R} $\varphi(\lambda - x) = c \varphi(\lambda + x)$, that is

$$\hat{\varphi}(-\xi) = c e^{2i\lambda\xi} \hat{\varphi}(\xi). \quad (2.16)$$

By (2.9) c should be 1 while using (2.8) and (2.16) with ξ and 2ξ , it follows that symmetry for φ is equivalent to

$$m_0(-\xi) = e^{2i\lambda\xi} m_0(\xi). \quad (2.17)$$

Since m_0 has real coefficients, $m_0(-\xi) = \overline{m_0(\xi)}$ and we can write $m_0(\xi) = a(\xi)e^{-i\lambda\xi}$, where $a(\xi) = e^{i\lambda\xi}m_0(\xi)$ takes real values. In conclusion, symmetry for φ is equivalent with linear phase for m_0 .

In terms of H , (2.17) becomes

$$H(z) = z^{2\lambda} H(z^{-1}). \quad (2.18)$$

Since replacing φ by an integer translate still generates the same Multiresolution Analysis, we can consider λ in the interval $[0, 1)$, and then the only possible values for λ are 0 or $1/2$.

Thus, either the scaling function is even (0-symmetry) or it is even about $1/2$ ($1/2$ -symmetry). By (2.4) and (2.18) the corresponding conditions on the QMF are

0-SYM

$$H^2(z) + H^2(-z) = 1 \quad (2.19)$$

$$H(z) = H(z^{-1}). \quad (2.20)$$

1/2-SYM

$$H^2(z) - H^2(-z) = z \quad (2.21)$$

$$H(z) = zH(z^{-1}). \quad (2.22)$$

We now show which kind of symmetry we impose on the wavelet if we start with a scaling function with symmetry $\lambda = s_\varphi$.

As in (2.16), we require

$$\hat{\psi}(-\xi) = ce^{2is_\varphi\xi} \hat{\psi}(\xi), \quad (2.23)$$

and then c is ± 1 because ψ has \mathbf{L}^2 norm equal to 1.

Using (2.10) and (2.11),

$$\hat{\psi}(2\xi) = \overline{m_0(\xi + \pi)} e^{-i\xi} d(2\xi) \hat{\varphi}(\xi)$$

where $d(\xi)$ is a rational trigonometric function and $d(\xi)d(-\xi) = 1$.

Because of (2.16) and (2.17)

$$\hat{\psi}(-2\xi) = \frac{e^{i\xi} \overline{m_0(-\xi - \pi)} \hat{\varphi}(-\xi)}{d(2\xi)} = \frac{e^{2i\xi} e^{-2\pi i s_\varphi} \hat{\psi}(2\xi)}{d^2(2\xi)}.$$

Hence, to obtain (2.23) we need $d(\xi) = \pm e^{im\xi}$ with $m \in \mathbf{Z}$ and $2s_\psi = 1 - 2m$. Choosing $d(\xi) = -1$, that is

$$m_1(\xi) = -e^{-i\xi} \overline{m_0(\xi + \pi)}, \quad (2.24)$$

we obtain $s_\psi = 1/2$, unique up to integer translations. Consequently, ψ is even about $1/2$ or odd about $1/2$ depending on whether s_φ is 0 or $1/2$. An example of the latter case is the Haar basis, where the wavelet and scaling function are $\psi = 1_{[0, \frac{2}{2})} - 1_{[\frac{2}{2}, 1)}$ and $\varphi = 1_{[0, 1)}$. Here 1_I is the indicator function of the interval I .

The choice of m_1 in (2.24) can be written in terms of the z -transforms of the filters as

$$G(z) = -zH(-z^{-1}). \quad (2.25)$$

3 The characterization of the 0-SYM case

First, we introduce some additional notations and definitions.

Sometimes we omit the variables of a function as in the following transformations

$$\tilde{f} = \tilde{f}(z) = f(-z) \quad (3.1)$$

$$f^* = f^*(z) = f(z^{-1}). \quad (3.2)$$

We use the notation s_t^f for the sign of the function f at the value t ,

$$s_t^f = \text{sign}(f(t)) \quad (3.3)$$

If E is a polynomial of degree d its *reciprocal* is $z^d E(z^{-1})$.

We call a an *all-pass* function if

$$a(z)a(z^{-1}) = 1. \quad (3.4)$$

Rational all-pass functions can be described as:

$$a(z) = \pm z^n \frac{E(z)}{E(z^{-1})} \quad (3.5)$$

where n is an integer, E is a polynomial coprime with its reciprocal and E does not vanish at zero.

The *order* of a rational function $\frac{P}{Q}$, where P and Q are coprime polynomials, is the maximum of their degrees.

We use \circ for composition of functions. If n is a positive integer, $f^{[n]}$ denotes $f \circ \dots \circ f$ (n times), $f^{[0]}$ is the identity and $f^{[-1]}$ is the inverse of f .

The bilinear transformation

$$\beta(z) = \frac{1-z}{1+z} \quad (3.6)$$

maps the unit disk onto the right half plane $\{z : \Re(z) > 0\}$ and the unit circle onto the imaginary axis:

$$\beta(e^{i\xi}) = -i \tan\left(\frac{\xi}{2}\right). \quad (3.7)$$

The transformation β gives a natural way to relate analog with digital filter design; see [21] and [16, Page 219], for more details. The conjugation by β of f is $f_\beta = \beta \circ f \circ \beta^{[-1]}$ and similarly for $\tilde{\beta}(z) = \beta(-z)$.

The next theorem provides an explicit description of all 0-SYM filters. Conjugation by β is the key element for the simplicity of this parameterization. In Corollary 2 below we explain how to obtain vanishing moments for the associated wavelet and scaling functions.

Theorem 1 *There is a bijection between the set of 0-SYM filters H and the set of even all-pass functions a given by:*

$$a \mapsto H = \beta \circ \frac{1}{(1 + \sqrt{2}a)^2} \circ \beta^{[-1]}, \quad (3.8)$$

$$H \mapsto a = \frac{1 + H(z) - H(-z)}{\sqrt{2}H(-z)} \circ \beta. \quad (3.9)$$

If $H(1) = 1$, H can be described as

$$H(z) = \frac{A(z)(A(z) \pm \sqrt{2}A(-z))}{A(z)^2 + A(-z)^2 \pm \sqrt{2}A(z)A(-z)}, \quad (3.10)$$

where $A(z)$ is a polynomial of even degree, coprime with $A(-z)$, and equal to its reciprocal. Also, $A(0)A(1) \neq 0$ and $A(-1) = 0$.

Proof The transformation β satisfies:

$$\beta^{[-1]} = \beta, \quad (3.11)$$

$$\beta^* = -\beta, \quad (3.12)$$

$$\tilde{\beta} = \frac{1}{\beta}. \quad (3.13)$$

Conjugation by β yields,

$$d \text{ is all-pass} \iff d_\beta \text{ is an odd function}, \quad (3.14)$$

$$d = d^* \iff d_\beta \text{ is an even function}. \quad (3.15)$$

Define the function ν as

$$\nu(z) = \beta \circ \frac{1}{(1 + \sqrt{2}z^{-1})^2} = \frac{1 + \sqrt{2}z}{1 + \sqrt{2}z + z^2}. \quad (3.16)$$

First we show that the maps in (3.8) and (3.9) are well defined. We need to check that for a even all-pass

$$H = \nu \circ a(z^{-1}) \circ \beta = \nu \circ \beta \circ \left(\frac{1}{a}\right)_\beta \quad (3.17)$$

belongs to 0-SYM. Because a is even, a_β equals $(a_\beta)^*$, which gives (2.20). On the other hand, (2.19) is invariant under odd functions and $(\frac{1}{a})_\beta$ is odd, hence we only need to verify that $\nu \circ \beta$ satisfies (2.19). This follows replacing z by $\beta(z)$ in the identity

$$\nu^2(z) + \nu^2(z^{-1}) = 1. \quad (3.18)$$

Next we start with H with 0-SYM and check that a in (3.9) is even and all-pass. Using (3.14) and (3.15) it suffices to show that

$$\beta \circ \frac{1 + H - \tilde{H}}{\sqrt{2}\tilde{H}}$$

is odd and invariant under z^{-1} . The latter follows from (2.20), while using (3.12) the former is equivalent to the identity

$$\beta \circ \frac{1 + \tilde{H} - H}{\sqrt{2}H} = \beta \circ \frac{\sqrt{2}\tilde{H}}{1 + H - \tilde{H}}$$

or

$$1 - (\tilde{H} - H)^2 = 2H\tilde{H}$$

which follows from (2.19).

To establish that the maps in (3.8) and (3.9) are each others inverses, let us denote them by \mathcal{S} and \mathcal{T} respectively.

Let a be an even all-pass function. Since

$$\mathcal{S}(a) = \nu \circ a^* \circ \beta$$

we have

$$\frac{1 + \mathcal{S}(a) - \mathcal{S}(a)(-z)}{\sqrt{2}\mathcal{S}(a)(-z)} = \left(\frac{1 + \nu(a^*) - \nu(a)}{\sqrt{2}\nu(a)}\right) \circ \beta.$$

Consequently, using that a is all-pass and the identity

$$\frac{1 + \nu^* - \nu}{\sqrt{2}\nu} = z,$$

we have

$$\mathcal{T} \circ \mathcal{S}(a) = \left(\frac{1 + \nu^* - \nu}{\sqrt{2}\nu}\right)(a) = a.$$

Finally, if H is a 0-SYM filter,

$$\begin{aligned}
1 + \sqrt{2}\mathcal{T}(H) &= \left(\frac{\tilde{H}}{1-H}\right) \circ \beta, \\
\mathcal{S} \circ \mathcal{T}(H) &= \beta \circ \left(\frac{(1-H)^2}{\tilde{H}^2}\right) = \frac{\tilde{H}^2 - (1-H)^2}{\tilde{H}^2 + (1-H)^2} \\
&= \frac{1 - H^2 - (1 + H^2 - 2H)}{1 - 2H + (H^2 + \tilde{H}^2)} = H.
\end{aligned}$$

For the second part of the theorem, we write the even all-pass function a as

$$a(z) = s_i^a(-1)^{m+r} \frac{B(z^2)}{z^{2m+2r} B(z^{-2})} \quad (3.19)$$

where $r \geq 0$, $m \geq 1$ (because $H(1) = 1$), $B(z) = \sum_{j=0}^r b_j z^j$ with $b_0 b_r \neq 0$, and B coprime with its reciprocal. In particular $B(1) \neq 0$.

Thus,

$$a^*(\beta(z)) = s_i^a(-1)^{m+r} \frac{A(-z)}{A(z)} \quad (3.20)$$

where

$$A(z) = (1+z)^{2m+2r} B(\beta^2(z)) \quad (3.21)$$

$$= (1+z)^{2m} \sum_{j=0}^r b_j (1-z)^{2j} (1+z)^{2r-2j}. \quad (3.22)$$

The properties for A follow from the conditions on B while (3.10) is the result of applying ν to (3.20).

■

As stated in the introduction, we can obtain an almost interpolating 0-SYM filter by simply choosing an associated all-pass function with a high number of poles at 0.

Corollary 2 *Consider a and H as in the previous theorem and let $m > 0$. The following conditions are equivalent:*

- 1** *The multiplicity of 0 as a pole of a is $2m$.*
- 2** *The multiplicity of -1 as a zero of H is $2m$.*
- 3** *The multiplicity of 1 as a zero of $1 - H$ is $4m$.*

If any of these conditions holds then by (2.14) and (2.15)

$$\begin{aligned}
\int_{\mathbf{R}} x^k \psi(x) dx &= 0 \quad \text{for } 0 \leq k < 2m, \quad \text{and} \\
\int_{\mathbf{R}} x^k \varphi(x) dx &= \delta_{k0} \quad \text{for } 0 \leq k < 4m,
\end{aligned}$$

where ψ and φ are the wavelet and scaling functions associated to H .

Proof We simply show that the three conditions are equivalent with the polynomial A of (3.22) having $2m$ zeroes at -1 . Part **1** follows from (3.19) and (3.22) while (3.10) gives Part **2** (note that $A(1) \neq 0$ but $A(1) = 0$ because $m > 0$). The equivalence with Part **3** holds because (3.10) yields

$$1 - H(z) = \frac{A(-z)^2}{A(z)^2 + A(-z)^2 \pm \sqrt{2}A(z)A(-z)} \quad (3.23)$$

where the polynomials in the numerator and denominator are coprime. ■

Remarks 3 1. We can use the theorem as a dictionary to reformulate properties of H in terms of a and vice versa. Let us list some of them:

$$z_0 \in H^{-1}\{1\} \iff a(\beta(z_0)) = \infty \quad (3.24)$$

$$z_0 \in H^{-1}\{-1\} \iff a(\beta(z_0)) = -\frac{1}{\sqrt{2}} \quad (3.25)$$

$$z_0 \in H^{-1}\{\infty\} \iff a(\beta(z_0)) = e^{\pm i\pi 5/4} \quad (3.26)$$

Because of symmetry and the QMF condition, the frequency response of H only takes values in the real interval $[-1, 1]$. In Table 1 we listed the correspondence between values of H on the unit circle and values of a on the imaginary axis. This correspondence can be easily obtained from (3.17).

2. Because of (2.19), the zeroes of H are the negatives of the preimages by H of 1 and -1 . These two sets of preimages are nicely represented in the numerator of (3.10). The zeroes of $A(-z)$ (counted twice) are exactly the set $H^{-1}\{1\}$ (see (3.23)) while the set $H^{-1}\{-1\}$ is given by the zeroes of $A(-z) \pm \sqrt{2}A(z)$ (counted twice) because

$$1 + H(z) = \frac{(A(-z) \pm \sqrt{2}A(z))^2}{A(z)^2 + A(-z)^2 \pm \sqrt{2}A(z)A(-z)}. \quad (3.27)$$

The even multiplicity of the preimages of 1 and -1 agrees with the fact that the frequency response of H cannot take values outside $[-1, 1]$.

3. The filter H cannot have poles either on the unit circle or on the real line, because β maps those values to the imaginary and real axes, and a , being even and real, maps them to the real line.

Also, if H were a polynomial, $H^{-1}\{\infty\} = \{\infty\}$, and we would have

$$a(-1) = e^{\pm i\pi 5/4},$$

leading to a contradiction because a has real coefficients. Therefore, we recover the result on the absence of real polynomial QMFs generating wavelets with even symmetry [20, 6].

4. Using the notation in (3.3), it follows from (3.19) that $s_1^a = (-1)^{m+r} s_i^a$.

Since $H(0)$ and $H(i)$ are real numbers, (2.19) implies

$$H(0) = \frac{s_0^H}{\sqrt{2}} \quad \text{and} \quad H(i) = m_0(\pi/2) = \frac{s_i^H}{\sqrt{2}}. \quad (3.28)$$

Finally, (3.9) gives

$$s_1^a = s_0^H \quad \text{and} \quad s_i^a = s_i^H. \quad (3.29)$$

$H(e^{i\xi})$	$a(-i \tan(\frac{\xi}{2}))$
<hr/>	
$\{-1\}$	$\{-\frac{1}{\sqrt{2}}\}$
$(-1, 0)$	$(-\sqrt{2}, -\frac{1}{\sqrt{2}}) \cup (-\frac{1}{\sqrt{2}}, 0)$
$\{0\}$	$\{-\sqrt{2}, 0\}$
$(0, 1)$	$(-\infty, -\sqrt{2}) \cup (0, +\infty)$
$\{1\}$	$\{\infty\}$

Table 1: Correspondence between values of H and a

We now identify a minimal set of parameters defining each 0-SYM filter.

Parameterization in terms of preimages of one

Assume $H(1) = 1$. From (3.24), the set of preimages of 1 counted with their multiplicity (the set $H^{-1}\{1\}$) completely determines the poles of the all-pass function a . Therefore, up to a sign, a , and thus H , are completely determined by $H^{-1}\{1\}$. We fix the sign by choosing s_1^a , that is s_1^H .

The conclusion is that two solutions of 0-SYM are the same if their sets of preimages of 1 and their values at i are identical.

We now study the degrees of freedom in the set $H^{-1}\{1\}$.

First, because of (3.23), $H^{-1}\{1\}$ is the set of zeroes of $A(-z)^2$. Using the properties of A in (3.10) and (3.22), $A(-z)$ has $2(m+r)$ zeroes, $2m$ of them being 1. Also, if w belongs to $H^{-1}\{1\}$, \bar{w} and w^{-1} belong to $H^{-1}\{1\}$ but $-w$ does not.

Therefore, once s_i^H and m are fixed, there are only r degrees of freedom for a 0-SYM filter of order $4(m+r)$. We isolate these r parameters in a subset Λ of $H^{-1}\{1\}$ by discarding reciprocals and the value 1,

$$\Lambda = \{\lambda_1, \dots, \lambda_r\}. \quad (3.30)$$

In the next theorem, we rewrite our previous parameterization of H to show the dependence on Λ .

Theorem 4 *Let H be any 0-SYM filter with $2m$ zeroes at -1 and Λ the subset of $H^{-1}\{1\}$ defined in (3.30). If ν is the function defined in (3.16) and η is the function*

$$\eta(z) = \frac{z + z^{-1}}{2}, \quad (3.31)$$

then

$$H(z) = \nu \left(s_i^H \beta(\eta(z))^m \prod_{j=1}^r \beta\left(\frac{\eta(z)}{\eta(\lambda_j)}\right) \right). \quad (3.32)$$

Proof Since (3.17) and (3.20) imply

$$H(z) = \nu\left(a\left(\frac{1}{\beta}\right)\right) = \nu(s_i^a(-1)^{m+r} \frac{A(-z)}{A(z)}),$$

it suffices to show that

$$s_i^a(-1)^{m+r} \frac{A(-z)}{A(z)} = u(\eta(z)) \quad (3.33)$$

where

$$u(z) = s_i^a \beta(z)^m \prod_{j=1}^r \beta\left(\frac{z}{\eta(\lambda_j)}\right). \quad (3.34)$$

We know that A has degree $2(m+r)$, it is equal to its reciprocal, and (from (3.23)) its zeroes are the negatives of the elements of the set Λ considered together with their inverses. Therefore, there exist a constant c and a polynomial E ,

$$E(z) = \prod_{j=1}^r (z + \eta(\lambda_j)),$$

such that

$$A(z) = cz^{m+r}(1 + \eta(z))^m E(\eta(z))$$

and the theorem follows. ■

We have now a straightforward way of designing a 0-SYM filter H by imposing the locations where the filter takes the value one.

Since the value $m'_0(\frac{\pi}{2})$ gives an idea of how narrow the transition band is, we will express it in terms of $H^{-1}\{1\}$. The next corollary implies that if we trade vanishing moments for any other value on the arc $(\frac{\pi}{2}, \pi)$ of the unit circle (the stopband), then the slope of m_0 is steeper at $\frac{\pi}{2}$.

Corollary 5 *Let H as in Theorem 4 and $m_0(\xi) = H(e^{-i\xi})$. Then*

$$m'_0\left(\frac{\pi}{2}\right) = -(2 - \sqrt{2}s_i^H)\left(m + \sum_{j=1}^r \frac{1}{\eta(\lambda_j)}\right). \quad (3.35)$$

Proof From the proof of Theorem 4, we have

$$m_0(\xi) = \nu(u(\cos(\xi))). \quad (3.36)$$

Taking derivatives at π and using $u(0) = s_I^H$ we obtain

$$m'_0\left(\frac{\pi}{2}\right) = -\nu'(s_I^H)u'(0) = -\frac{\sqrt{2} - 2s_I^H}{2}u'(0).$$

Since

$$\frac{\beta'(0)}{\beta(0)} = -2$$

the logarithmic derivative of (3.34) at 0 is,

$$\frac{u'(0)}{u(0)} = -2\left(m + \sum_{j=1}^r \frac{1}{\eta(\lambda_j)}\right). \quad \blacksquare$$

The next two results are obtained imposing conditions on the set $H^{-1}\{1\}$.

Proposition 6 *Let H be a 0-SYM filter with $s_I^H = 1$ and*

$H^{-1}\{1\} \cap \{z : |z| = 1\} = \{1\}$. Then

$$H(e^{i\xi}) > 0 \quad \text{for } \xi \in (-\pi, \pi).$$

Proof Since $H(1) = 1$ and $H(z^{-1}) = H(z)$ it suffices to consider ξ in $(0, \pi)$. From (3.19),

$$a(\beta(e\xi)) = a(-i \tan(\frac{\xi}{2})) = \frac{E(x)}{x^{m+r}E(x^{-1})}, \quad (3.37)$$

where $x = \tan^2(\frac{\xi}{2})$, $m \geq 1$, and $E(x) = B(-x)$.

The zeroes of E are $-\beta(-\lambda)^2$ where $\lambda \neq 1$ belongs to $H^{-1}\{1\}$. (See (3.21) and the second remark in Remarks 3.) These zeroes cannot be positive numbers because

$$-\beta(-\lambda)^2 \in (0, +\infty) \iff \beta(-\lambda) \in \mathbf{R}i \iff |\lambda| = 1,$$

which contradicts our assumption on $H^{-1}\{1\}$.

Thus, E has constant sign in $(0, +\infty)$ and (3.37) implies that $a(\beta(e\xi))$ is positive for $\xi \in (0, \pi)$. The result follows using Table 1. \blacksquare

We now discuss a typical filter design problem. Suppose the desired frequency response vanishes at certain specific locations of the stopband. By choosing appropriate preimages of 1 in the passband, we can obtain the given zeroes but, in principle, we cannot avoid further zeroes in the stopband because they are related to preimages of -1 . The next theorem provides sufficient conditions to prevent this

last situation from happening and, incidentally, forces all poles of the filter to be purely imaginary. By restricting $H^{-1}\{1\}$ to the right half plane we can parameterize 0-SYM filters with any preassigned set of zeroes in the stopband. We can then obtain a filter with minimal order or, considering more parameters, even impose additional properties.

Theorem 7 *Let H be any 0-SYM filter with $H^{-1}\{1\} \subseteq \{z : \Re(z) > 0\}$. We have the following properties:*

A $H^{-1}\{\infty\} \subseteq \mathbf{R}i,$

B $H^{-1}\{-1\} \subseteq \{z : \Re(z) < 0\}.$

Proof The assumption on H and (3.24) imply that all poles of the corresponding all-pass function a are in the unit disk. Therefore, $a(z^{-1}) = \frac{1}{a(z)}$ is a finite Blaschke product and then a maps the unit circle into itself and $\{z : |z| > 1\}$ into the unit disk and vice versa.

To verify Part **A** consider $p \in H^{-1}\{\infty\}$. By (3.26), $a(\beta(p))$ and then $\beta(p)$ belong to the unit circle. Thus, p is purely imaginary by (3.7).

For Part **B** start with $p \in H^{-1}\{-1\}$. Since (3.25) implies that $a(\beta(p))$ belongs to the unit disk, we have $|\beta(p)| > 1$ and then $\Re(p) < 0$. ■

Corollary 8 *For $\{\theta_1 \cdots \theta_s\}$ in $(\frac{\pi}{2}, \pi)$ let*

$$H(z) = \nu \left(s_i^H \beta(\eta(z))^m \prod_{k=1}^t \beta\left(-\frac{\eta(z)}{\cos(\theta_k)}\right) \prod_{j=1}^s \beta\left(\frac{\eta(z)}{\eta(\lambda_j)}\right) \right), \quad (3.38)$$

where the set $\{\lambda_1, \dots, \lambda_t\}$ is contained in $\{z : \Re(z) > 0 \text{ and } |z| \neq 1\}$ and satisfies the conditions listed for (3.30).

Then, H is a 0-SYM filter that vanishes in the stopband only at the $2(m+s)$ values

$$\left\{ e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_s}, e^{-i\theta_s}, \underbrace{-1, \dots, -1}_{2m} \right\}.$$

Proof Using Theorem 4, define H with $2m$ zeroes at -1 and $\Lambda = \{-e\theta_1, \dots, -e\theta_s, \lambda_1, \dots, \lambda_t\}$. Theorem 7 assures that there are no other zeroes in the stopband. ■

Note that “wrong” choices of $H^{-1}\{1\}$ can lead to very poor frequency responses. In Figure 1 we plotted the frequency responses of two order 12 filters with $s_i^H = 1$ and two zeroes at -1 but different choices for the other zeroes. Their set of preimages, zeroes, and poles are listed in Table 2. In one case we have chosen the zeroes in order to violate Cohen’s condition. The poor frequency response is not

a peculiarity of this choice. Perturbing the values of the preimages we still obtain a similar response. Note that the zeroes either belong to the unit circle or they are real numbers. For the other example, all preimages belong to the right half plane and therefore all zeroes related to preimages of -1 belong to the same half plane. In agreement with the previous theorem, all poles are purely imaginary.

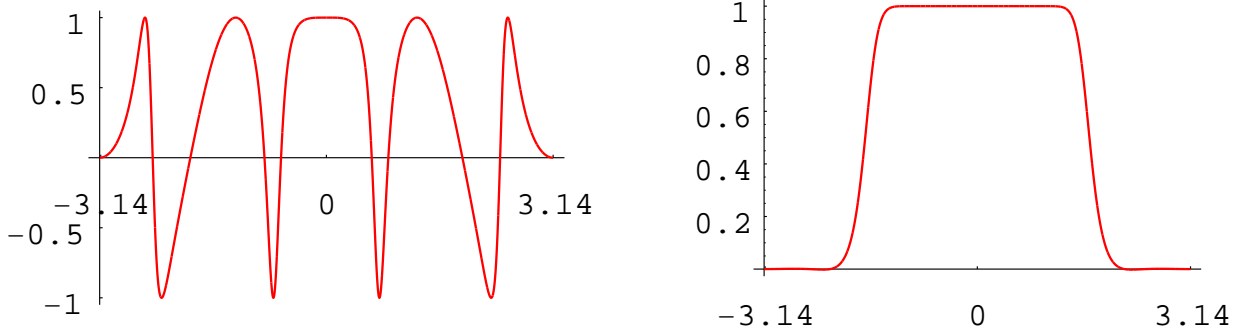


Figure 1: Two 0-SYM filters of order 12. On the left we violated Cohen's condition. On the right we choose all preimages of 1 in the right half plane.

4 Maximally Flat Filters

As in Daubechies' construction for compactly supported wavelets [6], we use our representation of the 0-SYM class to construct, for each fixed order of the filter, symmetric wavelets with maximum number of vanishing moments. For the reasons explained before, they correspond to the choice $a(z) = s_i^H(-1)^n z^{-2n}$ in (3.17). For simplicity of notation, we write $s_i^H = (-1)^\delta$, where δ is a positive integer.

4.1 Description and Properties

For δ and n positive integers, we define

$$E_{4n}^\delta(z) = \nu((-1)^{\delta+n} \beta^{2n}(z)) \quad (4.1)$$

that is

$$E_{4n}^\delta(z) = \frac{(1+z)^{2n}((1+z)^{2n} + (-1)^{\delta+n} \sqrt{2}(1-z)^{2n})}{(1+z)^{4n} + (1-z)^{4n} + (-1)^{\delta+n} \sqrt{2}(1-z^2)^{2n}}. \quad (4.2)$$

The filters E_{4n}^δ have maximum number of zeroes at -1 ($2n$ in fact) in the set of rational QMF with 0-SYM and order at most $4n$. For their frequency response, we use (3.7) and (4.1):

$$E_{4n}^\delta(e^{i\xi}) = \nu((-1)^\delta \tan^{2n} \frac{\xi}{2}) \quad (4.3)$$

$$= 1 - \frac{1}{\frac{1}{2} + (\frac{1}{\sqrt{2}} + (-1)^\delta \cot^{2n} \frac{\xi}{2})^2}. \quad (4.4)$$

It readily follows that, as n goes to ∞ , E_{4n}^δ is approaching the ideal low-pass filter ι

$$\iota(\xi) = \begin{cases} 1 & \text{if } 0 \leq |\xi| < \pi/2, \\ 0 & \text{if } \pi/2 < |\xi| < \pi. \end{cases} \quad (4.5)$$

When $\delta = 0$ the convergence is uniform because ν is a decreasing function in the interval $(0, 1)$. To illustrate the convergence, a few filters are simultaneously plotted in Figure 2 (for $\delta = 0$) and Figure 3 (for $\delta = 1$).

Because of (2.25), the filters m_1 associated with the wavelet converge to the ideal high-pass filter. Compare this situation with Daubechies' family of maximally flat filters that converge to the ideal filters only in absolute value [11].

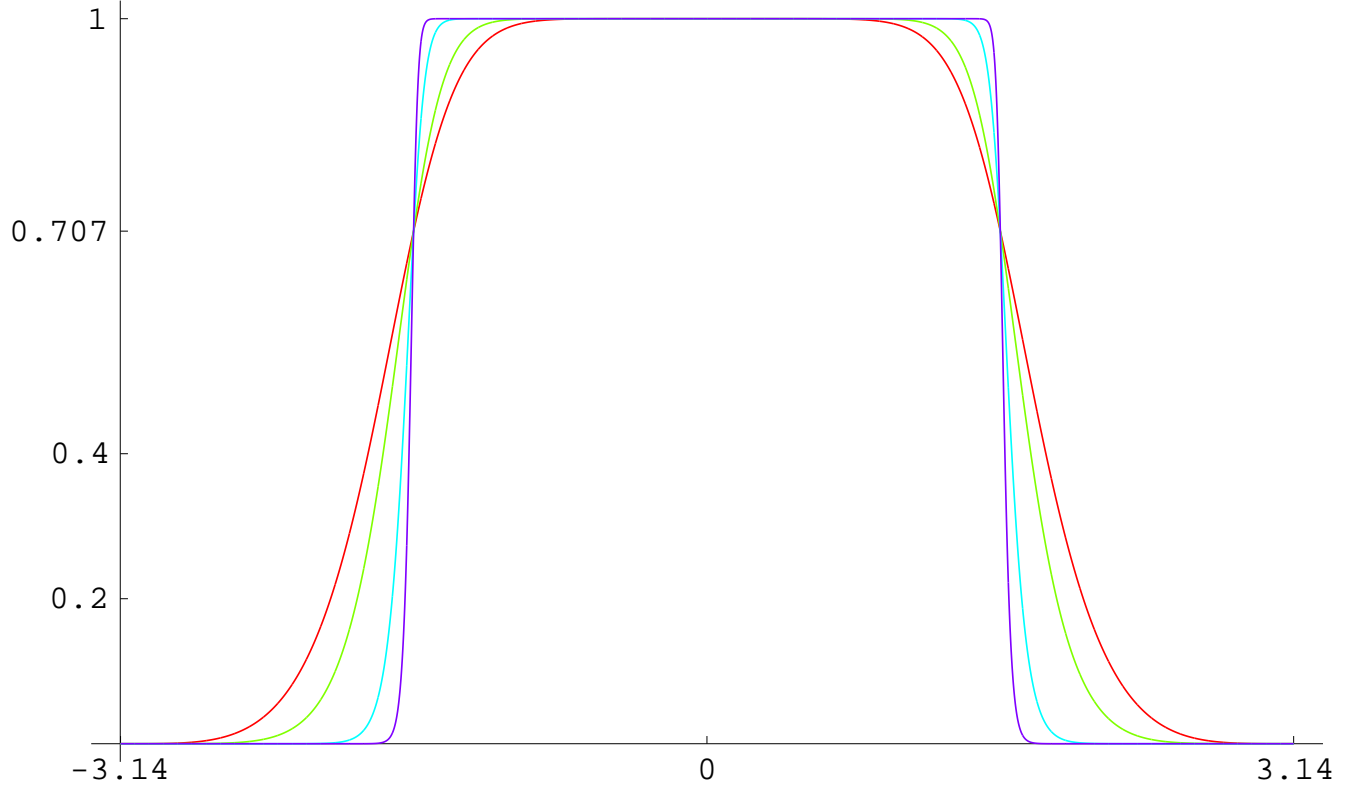


Figure 2: Approximating the ideal low-pass filter by maximally flat filters E_{4n}^0 . The filters shown in the picture correspond to $n = 2, 3, 8$, and 20 .

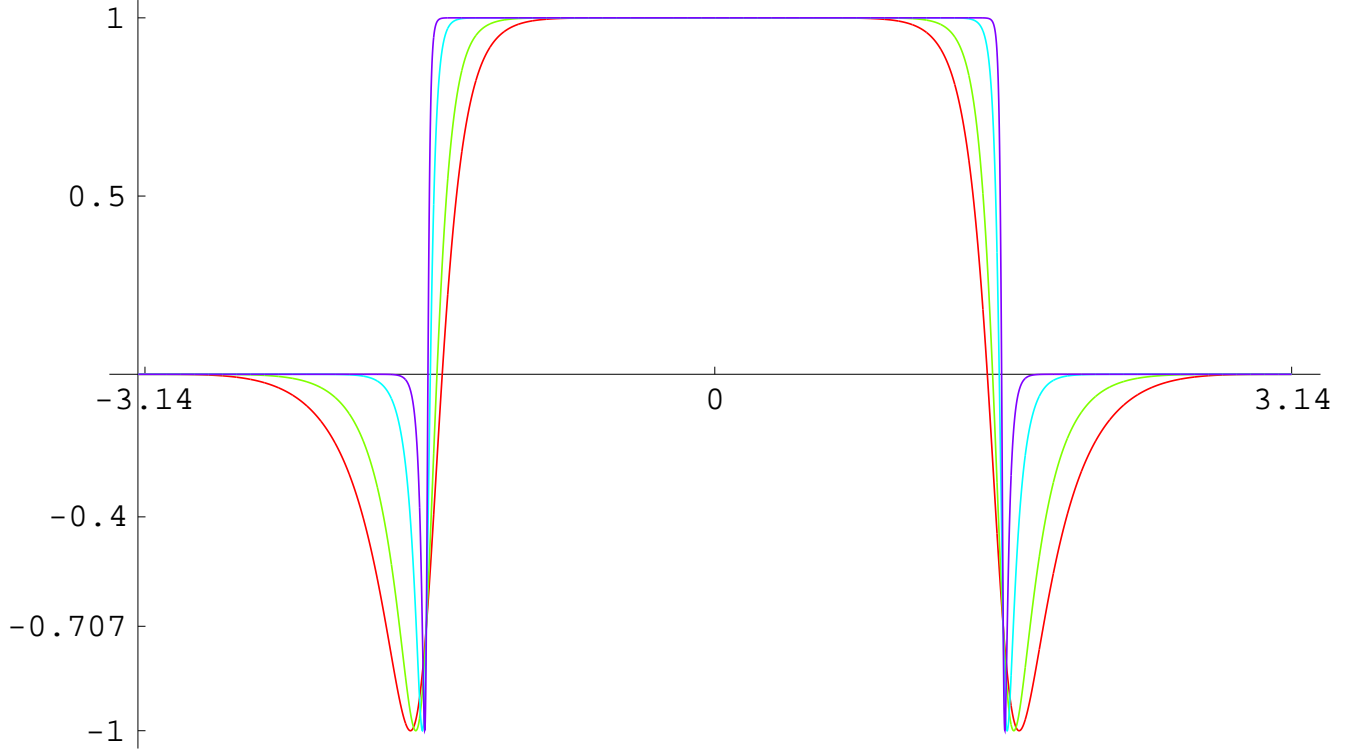


Figure 3: Approximating the ideal low-pass filter by maximally flat filters E_{4n}^1 . The filters shown in the picture correspond to $n = 2, 3, 8$, and 20 .

Next, we explicitly find all zeroes and poles of E_{4n}^δ . In agreement with the results of the previous section, the frequency response of E_{4n}^0 is positive in $(-\pi, \pi)$ (Proposition 6), all poles of E_{4n}^δ are purely imaginary (Theorem 7, Part **A**), and except for -1 , all zeroes of E_{4n}^δ belong to the right half plane (Theorem 7, Part **B**). The latter result also holds for Daubechies' family of maximally flat filters [19, Theorem 2].

Zeroes of E_{4n}^δ

We already know that -1 is a zero of multiplicity $2n$. The other zeroes are the values of ω such that $H(-\omega) = -1$. Since $a(z) = (-1)^{\delta+n} z^{-2n}$, using (3.25) we need to solve

$$\beta^{2n}(\omega) = \frac{e^{i\pi(n+\delta-1)}}{\sqrt{2}}.$$

It follows that the zeroes are

$$\beta \left(2^{-\frac{1}{4n}} e^{\frac{i\pi}{2n}(2j+n+\delta-1)} \right), \quad (4.6)$$

for $-n < j \leq n$. Note that they belong to the right half-plane.

With respect to zeroes on the unit circle, we now show that there is a pair of complex conjugate zeroes if $\delta = 1$ and no zeroes if $\delta = 0$.

We check in (4.6) when the argument of β is purely imaginary. Since $-n < j \leq n$, it follows that $\sin(\frac{\pi}{2n}(2j + \delta - 1)) = 0$ only if $\delta = 1$ and either $j = 0$ or $j = n$. In either case, the real part equals

$$\frac{\gamma}{\eta}(2^{\frac{1}{4n}}) = \beta(2^{-\frac{1}{2n}}).$$

For all n the numbers above are less than $\frac{1}{2}$ and Cohen's condition holds.

Note that for any real numbers λ and ξ ,

$$\beta\left(\frac{e^{i\xi}}{\lambda}\right) = \frac{\gamma(\lambda) - \sin \xi \, i}{\eta(\lambda) + \cos \xi} \quad (4.7)$$

where η is the function in (3.31) and $\gamma(z) = \frac{z - z^{-1}}{2}$.

Thus, except for -1 , the zeroes of E_{4n}^δ can be also expressed as

$$\left\{ \frac{\gamma(2^{\frac{1}{4n}}) - i \cos(\frac{\pi}{2n}(2j + \delta - 1))}{\eta(2^{\frac{1}{4n}}) - \sin(\frac{\pi}{2n}(2j + \delta - 1))} \right\}, \quad (4.8)$$

for $-n < j \leq n$.

Poles of E_{4n}^δ

By (3.26), we need the values of z such that

$$\beta^{2n}(z) = e^{\pm i\pi(5/4 + \delta + n)}.$$

Because of (3.7) and (3.11) the poles are

$$\left\{ \pm i \tan \left(\frac{\pi}{16n} (5 + 8j + 4\delta + 4n) \right) \right\}$$

for $-n < j \leq n$.

Remark 9 The recurrence relation

$$E_{8s}^\delta = \nu((-1)^\delta \beta^{4s}) = E_{4s}^{\delta+s}\left(\frac{1}{\eta}\right),$$

follows substituting z by $\frac{1}{\eta}$ in (4.1) and using the identity

$$\beta \circ \frac{1}{\eta} \circ \beta^{[-1]} = z^2. \quad (4.9)$$

Thus, for all s ,

$$E_{2^n(4s)}^\delta = E_{4s}^{\delta+s}((1/\eta)^{[n]}),$$

and the behavior of E_{4n}^δ for large n is dictated by the iteration of the map $1/\eta$. This map is conjugate to z^2 implying that its Julia set is the unit circle. We have then another explanation why these families

are approaching the ideal filter: When n goes to ∞ , $(1/\eta)^{[\infty]}$ restricted to the unit circle has only three possible values. In fact,

$$(1/\eta)^{[\infty]}(e^{i\xi}) = \begin{cases} 1 & \text{if } |\xi| < \pi/2, \\ \infty & \text{if } |\xi| = \pi/2, \\ 0 & \text{if } \pi/2 < |\xi| \leq \pi. \end{cases} \quad (4.10)$$

Theorem 1 implies that

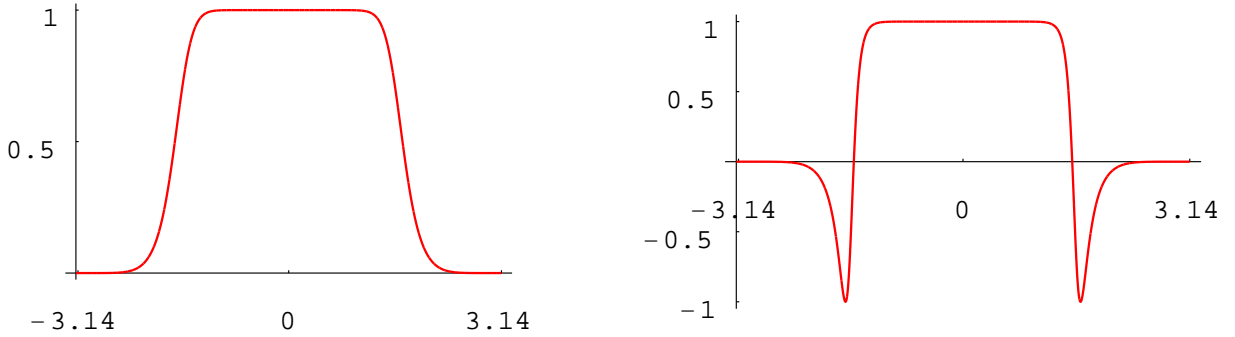
$$H_{\beta} = \beta \circ H \circ \beta^{[-1]} = \frac{1}{(1 + \sqrt{2}a)^2}$$

for any 0-SYM filter H . When $H(1) = 1$, this conjugation changes the location of the fixed point from 1 to 0. We can use $\tilde{\beta} = \beta(-z)$ to change it from 1 to ∞

$$H_{\tilde{\beta}}(z) = (1 + \sqrt{2}a(z^{-1}))^2.$$

The choice $a(z) = (-1)^{\delta+n}z^{-2n}$ yields the maximal family E_{4n}^{δ} whose set of preimages of 1 equals $\underbrace{\{1, \dots, 1\}}_{4n}$. Thus, the conjugation by $\tilde{\beta}$ of each E_{4n}^{δ} gives a polynomial! It is

$$(1 + (-1)^{\delta+n}\sqrt{2}z^{2n})^2. \quad (4.11)$$



For $\delta = 0, 1$

$$E_{12}^{\delta}(e^{i\xi}) = \frac{1 + (-1)^{\delta+1}\sqrt{2}\tan^6(\xi/2)}{1 + (-1)^{\delta+1}\sqrt{2}\tan^6(\xi/2) + \tan^{12}(\xi/2)}$$

have maximal number of zeros at π inside the set of rational linear phase QMFs of order 12.

Figure 4: Frequency responses of the maximal filters E_{12}^0 (left) and E_{12}^1 (right).

4.2 FIR implementation

Following [1], any IIR filter $H(z) = \frac{P(z)}{Q(z)}$ can be approximated on the unit circle by a FIR filter $F(z)$,

$$F(z) = z^{-N}P(z)\prod_{k=0}^n F_k(z^{2^k}). \quad (4.12)$$

The polynomials $F_k(z)$ only depend on the zeroes of $Q(z)$ and on the accuracy sought in the approximation.

$F(z)$ is not an exact QMF anymore, but if for w on the unit circle

$$\frac{|H(w) - F(w)|}{|H(w)|} \leq \epsilon,$$

then, for all z

$$F(z)F(z^{-1}) + F(-z)F(-z^{-1}) = 1 + R(z)$$

where the FIR filter $R(z)$ satisfies $|R(w)| < 3\epsilon$ for w on the unit circle. We assumed $\epsilon < \frac{1}{2\deg Q}$.

As an example, we approximate E_{12}^0 , the positive maximally flat 0-SYM filter with six zeroes at -1 . (See Figure 4.) To avoid negative powers of z in (4.12), we shift the phase of the filter by the positive integer N and approximate $z^N E_{12}^0(z)$.

The coefficients of the factors of the approximating filter for $\epsilon = 10^{-8}$ are listed in Table 3. The factor F_0 equals 1 because the denominator of E_{12}^0 is an even function. The wavelet and scaling functions generated with the approximating filter are plotted in Figures 5 and 6.

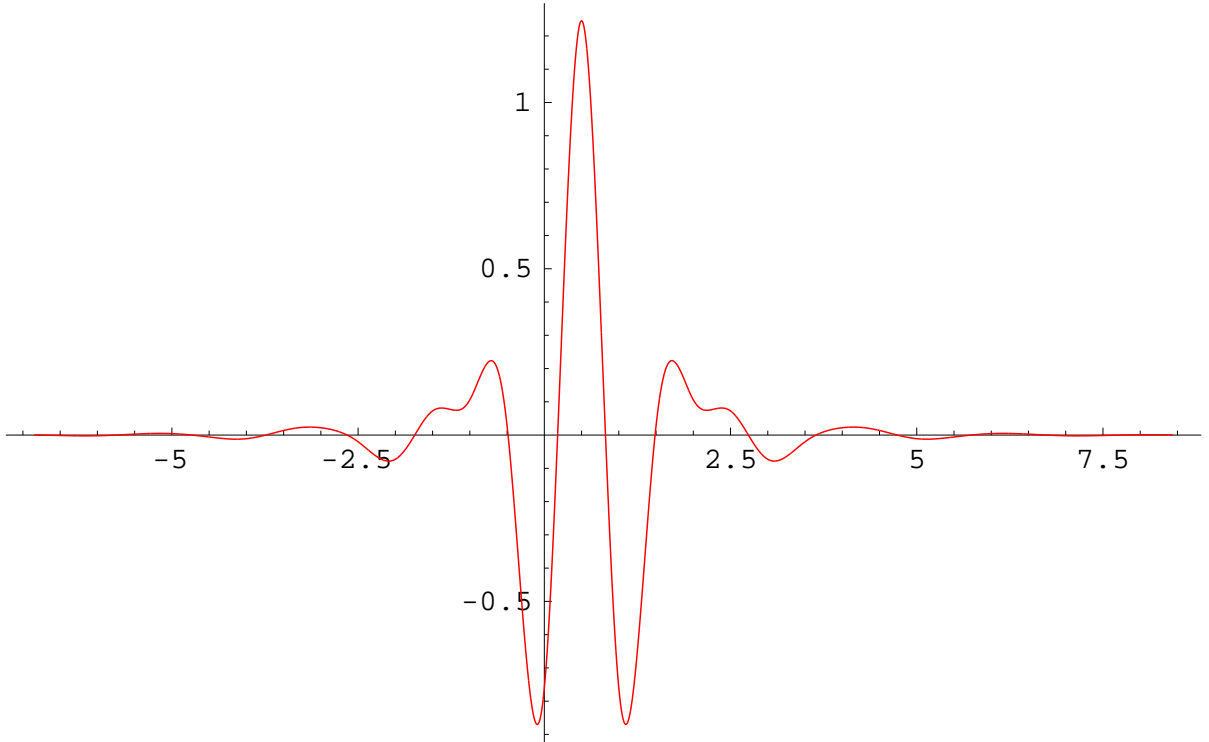


Figure 5: Wavelet function for FIR approximation of the maximally flat filter E_{12}^0 .

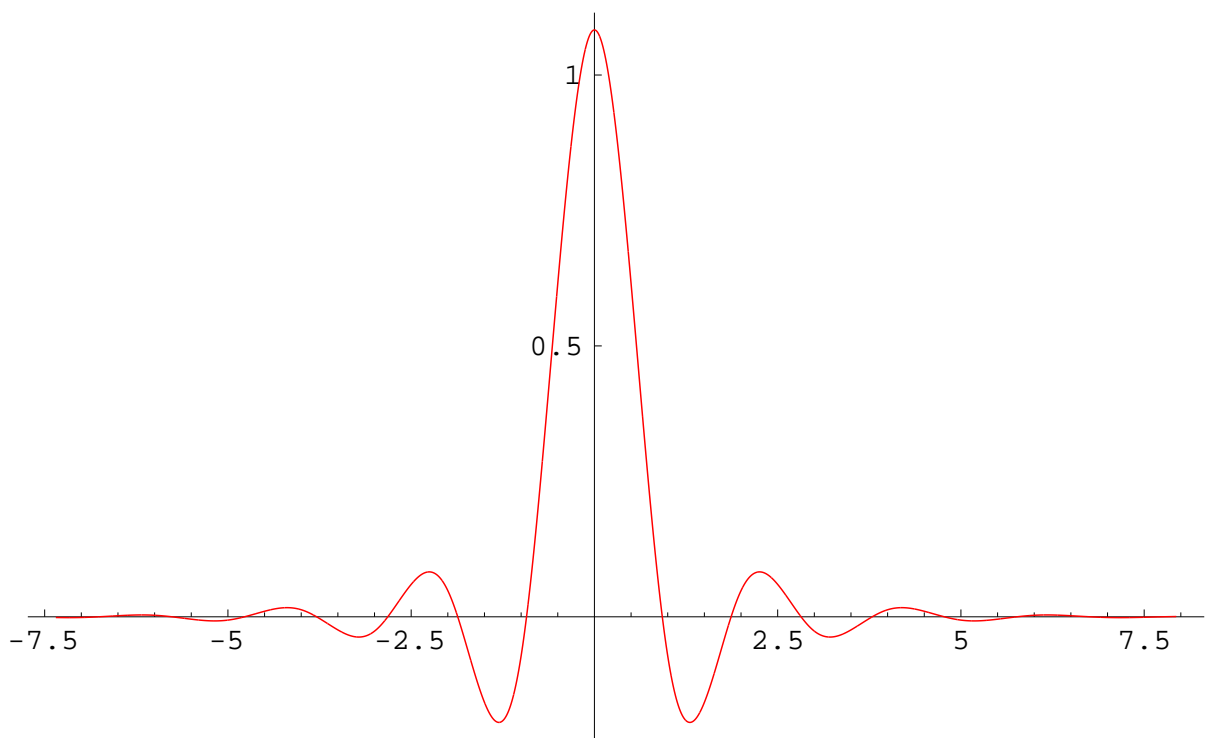


Figure 6: Scaling function for FIR approximation of the maximally flat filter E_{12}^0 .

Parameters	Zeroes	Poles
$s_i^H = m = 1$ $\Lambda = \{e^{\frac{2}{5}\pi i}, e^{\frac{4}{5}\pi i}\}$	-1	-0.84955807 - 0.74802903 I
	-1	-0.84955807 + 0.74802903 I
	-0.74212 - 0.67026705 I	-0.66304573 - 0.58380642 I
	-0.74212 + 0.67026705 I	-0.66304573 + 0.58380642 I
	-0.30901699 - 0.95105652 I	-0.40197132 I
	-0.30901699 + 0.95105652 I	0.40197132 I
	0.17142917	-2.4877396 I
	0.65396257 - 0.75652691 I	2.4877396 I
	0.65396257 + 0.75652691 I	0.66304573 - 0.58380642 I
	0.80901699 - 0.58778525 I	0.66304573 + 0.58380642 I
	0.80901699 + 0.58778525 I	0.84955807 - 0.74802903 I
	5.8333128	0.84955807 + 0.74802903 I
$s_i^H = m = 1$ $\Lambda = \{e^{0.21\pi i}, e^{0.31\pi i}\}$	-1	-0.083442717 I
	-1	0.083442717 I
	-0.79015501 - 0.61290705 I	-0.57528543 I
	-0.79015501 + 0.61290705 I	0.57528543 I
	-0.56208338 - 0.82708057 I	-0.73702991 I
	-0.56208338 + 0.82708057 I	0.73702991 I
	0.03560146 - 0.65573566 I	-1.356797 I
	0.03560146 + 0.65573566 I	1.356797 I
	0.036837087	-1.7382676 I
	0.082552825 - 1.5205228 I	1.7382676 I
	0.082552825 + 1.5205228 I	-11.984269 I
	27.146554	11.984269 I

Table 2: Zeroes and poles of filters constructed via their preimages of one.

Factor	Coefficients	Factor	Coefficients
P	-0.0001011263580012439	F_2	1.348299677989997e-7
	0.0029296875		0.007308809891655256
	0.01818488314800746		0.162081739736554
	0.0537109375		0.6612186310836452
	0.1156706046299813		0.162081739736554
	0.193359375		0.007308809891655256
	0.2324912771600248		1.348299677989997e-7
	0.193359375	F_3	0.0001276629992294306
	0.1156706046299813		0.03971627520745388
	0.0537109375		0.920312123586633
	0.01818488314800746		0.03971627520745388
	0.0029296875		0.0001276629992294306
	-0.0001011263580012439	F_4	1.925310635034675e-8
F_1	-0.00268082617584078		0.001585818961857287
	0.6429247852752233		0.996828323570073
	-4.433610674839401		0.001585818961857287
	8.58673343148004		1.925310635034675e-8
	-4.433610674839401	F_5	2.492072168636633e-6
	0.6429247852752233		0.999995015855663
	-0.00268082617584078		2.492072168636633e-6

Table 3: FIR approximation of a shifted maximal filter with six zeroes at -1 . The coefficients listed correspond to the factors in Eq. (4.12). In this example $N = 0$, $n = 5$, and $F_0 = 1$.

5 Conclusion

The results described in this paper allows one to explicitly construct all possible linear phase IIR quadrature mirror filters, when the slope of the phase is zero. Each filter is described using a minimal set of parameters which are directly related with the zeroes of the filter. This property permits to construct filters with any preassigned set of zeroes in the stopband. In particular, we construct, for each fixed order, the maximally flat filters which generate wavelets with maximal number of vanishing moments. We also give sufficient conditions on the parameters to assure that the frequency responses of the filters are always positive or that the filters have all their poles on the imaginary axis. We indicate how to implement these IIR filters by approximating them by cascades of FIR filters. The FIR approximation does satisfy the quadrature mirror condition with any desired accuracy. The factors in the approximation directly depend on the location of the poles of the IIR filter, what allow one to introduce the accuracy as part of the filter design process.

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